A NOTE ON SOME NEW FINITE DIVISION RING PLANES

BY

R. SANDLER

In [1; 3], certain finite nonassociative division algebras are defined and their autotopism groups determined. In this note, we generalize the definition of [1] and show that some of the new algebras are not isotopic with any of the old algebras. Thus, some new projective planes coordinatized by division rings will appear, although the new planes are very similar in collineation group structure with the planes coordinatized by the old algebras.

Let K be the finite field of p^m elements, p prime, $K=GF(p^m)$. Then K is cyclic over $F=GF(p)$, and the automorphism T defined by

$$xT = x^p \tag{1}$$

generates the automorphism group of K over F. Furthermore, $T^m=I$. If x, y are in K, we shall write

$$xy = xR(y) = yL(x). \tag{2}$$

We shall define a class of algebras as follows: Let k be an integer $1 \leqq k < m$ and let $S=T^k$. The algebra $\Omega_{\delta,k}$ for $\delta \neq 0$ in K is the vector space direct sum

$$K + \lambda K$$

of two copies of K. Multiplication is defined by

$$(x + \lambda y)(u + \lambda v) = [xu + \delta(yS)v] + \lambda[yu + (xS)v] \tag{3}$$

for x, y, u, v in K. If $L^{\delta,k}_{x+\lambda y}$ is taken to represent left multiplication in $\Omega_{\delta,k}$, then by (3), we have

$$L^{\delta,k}_{x+\lambda y} = \begin{pmatrix} R(x) & R(y) \\ R[\delta(yS)] & R[xS] \end{pmatrix}. \tag{4}$$

Now, $\Omega_{\delta,k}$ will be a *division* algebra if and only if $L^{\delta,k}_z$ is nonsingular for every $z \neq 0$ in $\Omega_{\delta,k}$; i.e., if and only if $x(xS)-\delta y(yS) \neq 0$ unless $x=y=0$. Thus $\Omega_{\delta,k}$ is a division algebra if and only if $\delta \neq c(cS)$ for any $c \in K$. But let ω be a primitive element of K, and let $\omega^\nu=\delta$. Then $\delta \neq c(cS)$ if and only if $\omega^\nu \neq \omega^{(1+p^k)\mu}$ for any μ where $\omega^\mu = c$. This condition is equivalent to $\mu(1+p^k) \not\equiv \nu (\mathrm{mod}\ p^m-1)$ for any μ, which happens if and only if $(1+p^k, p^m-1)$ does not divide ν.

In [1], the above algebras were studied in the case where k divides m, and the autotopism groups were determined. It will now be shown that the

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above defined class of algebras constitutes a true generalization and that new finite projective planes arise as a result.

The algebras Ω_{δ,k_1} and Ω_{ϵ,k_2} are said to be *isotopic* if there exist three nonsingular, linear transformations of Ω_2 onto Ω_1, Q, P, U, such that

$$PL_{xQ}^{\delta,k_1} = L_x^{\epsilon,k_2}U \qquad \text{for all } x \text{ in } \Omega_2. \tag{5}$$

The significance of the isotopy question is due to the theorem [2, Theorem 6] that two division algebras coordinatize isomorphic projective planes if and only if the algebras are isotopic.

To study the isotopy question, we need to know something about the middle nuclei of the algebras under consideration. The *middle nucleus* N_m of Ω is the set of all $a \in \Omega$ such that

$$(x \circ a) \circ y = x \circ (a \circ y) \tag{6}$$

for all x, $y \in \Omega$. It is known (see [3]) that the middle nucleus of any $\Omega_{\delta,k}$ is isomorphic with K; i.e., is the set of all $x + \lambda \cdot 0$, $x \in K$. Finally, the following result will be needed (see [4] for a proof): Let Ω_1, Ω_2 be two isotopic division algebras. For any c in the middle nucleus of Ω_2, there is an element a in the middle nucleus of Ω_1 such that

$$P\overset{1}{L}_a = \overset{2}{L}_c P, \tag{7}$$

where L^1 and L^2 represent left multiplication in Ω_1 and Ω_2, respectively.

Let $S_1 = T^{k_1}$, $S_2 = T^{k_2}$. Assume now that Ω_{δ,k_1} and Ω_{ϵ,k_2} are isotopic. Then there exists a triple Q, P, U, satisfying (5). The mapping P can be written in the form

$$P = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \tag{8}$$

where every H_{ij} is a linear transformation of K over $GF(p)$; i.e.,

$$H_{ij} = R(\overset{0}{a}_{ij}) + TR(\overset{1}{a}_{ij}) + T^2R(\overset{2}{a}_{ij}) + \cdots + T^{m-1}R(\overset{m-1}{a_{ij}}), \tag{9}$$

$\overset{k}{a}_{ij} \in K$. We can rewrite (7), then, as

$$\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & (aS_1) \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & (cS_2) \end{pmatrix} \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}. \tag{10}$$

But (10) is equivalent to the relations:

$$\begin{aligned} H_{11}R(a) &= R(c)H_{11}, \\ H_{12}R(aS_1) &= R(c)H_{12}, \\ H_{21}R(a) &= R(cS_2)H_{21}, \\ H_{22}R(aS_1) &= R(cS_2)H_{22}. \end{aligned}$$

But this implies [1, Lemma 2] that each $H_{ij}=U_{ij}R(h_{ij})$ where U_{ij} is an automorphism of K, and h_{ij} an element of K. Setting $H_{11}=VR(h_{11})$, we can write

$$\begin{aligned} H_{12} &= VS_1R(h_{12}), \\ H_{21} &= VS_2^{-1}R(h_{21}), \\ H_{22} &= VS_1S_2^{-1}R(h_{22}). \end{aligned} \tag{11}$$

Now in (5), let $x=1$, and see that $U=P\overset{1}{L}_{1Q}$. Thus (5) is equivalent to

$$P\overset{1}{L}_{xQ} = \overset{2}{L}_x P \overset{1}{L}_{1Q}. \tag{12}$$

Writing $1Q=u_0+\lambda u_1$ and using (11), we can write (12) as

$$\begin{pmatrix} VR(h_{11}) & VS_1R(h_{12}) \\ VS_2^{-1}R(h_{21}) & VS_1S_2^{-1}R(h_{22}) \end{pmatrix} \begin{pmatrix} a_0 & a_1 \\ \delta(a_1S_1) & (a_0S_1) \end{pmatrix}$$
$$= \begin{pmatrix} x_0 & x_1 \\ \epsilon(x_1S_2) & (x_0S_2) \end{pmatrix} \begin{pmatrix} VR(h_{11}) & VS_1R(h_{12}) \\ VS_2^{-1}R(h_{21}) & VS_1S_2^{-1}R(h_{22}) \end{pmatrix} \begin{pmatrix} u_0 & u_1 \\ \delta(u_1S_1) & (u_0S_1) \end{pmatrix},$$

where $(x_0+\lambda_0x_1)Q=a_0+\lambda a_1$ for $x_0+\lambda_0x_1\in\Omega_{\epsilon,k_2}$. Multiplying, we obtain:

$$\begin{aligned} VR(h_{11}a_0) + VS_1R(h_{12}\delta(a_1s_1)) &= VR[(x_0V)h_{11}u_0] + VS_1R[(x_0VS_1)h_{12}\delta(u_1S_1)] \\ &\quad + VS_2^{-1}R[(x_1VS_2^{-1})h_{21}u_0] + VS_1S_2^{-1}R[(x_1VS_1S_2^{-1})h_{22}\delta(u_1S_1)]; \end{aligned} \tag{a}$$

$$\begin{aligned} VR(h_{11}a_1) + VS_1R(h_{12}(a_0S_1)) &= VR[(x_0V)h_{11}u_1] + VS_1R[(x_0VS_1)h_{12}(u_0S_1)] \\ &\quad + VS_2^{-1}[(x_1VS_2^{-1})h_{21}u_1] + VS_1S_2^{-1}[(x_1VS_1S_2^{-1})h_{22}(u_0S_1)]. \end{aligned} \tag{b}$$

Now, the case $S_1=S_2$ has essentially been studied in [1], so the assumption is made here that $S_1\neq S_2$. Also, $S_1\neq I\neq S_2$. Thus we obtain, by equating distinct automorphisms on both sides of the equality signs,

$$(x_1VS_1S_2^{-1})\delta h_{22}(u_1S_1) = 0,$$

$(x_1VS_1S_2^{-1})h_{22}(u_0S_1)=0$, all $x_1\in K$. Since $\delta\neq 0$, $h_{22}(u_1S_1)=h_{22}(u_0S_1)=0$. Since everything in sight must be nonsingular, $u_0=u_1=0$ cannot hold. Hence, $h_{22}=0$. If we further assume that $S_1\neq S_2^{-1}$, we can write

$$(x_1VS_2^{-1})h_{21}u_0 = 0 = (x_1VS_2^{-1})h_{21}u_1$$

for all x_1 in K. Hence $h_{21}u_0=h_{21}u_1=0$. But this implies that either $u_0=u_1=0$ or $h_{21}=0$. In the former case, $1Q=0$; in the latter, P is singular since $h_{22}=0$, a contradiction. Hence, we may conclude that if Ω_{δ,k_1} and Ω_{ϵ,k_2} are isotopic and $S_1\neq S_2$, then $S_1=S_2^{-1}$.

On the other hand, if $S_1=S_2^{-1}$, the equation

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_0 & a_1 \\ \delta(a_1S_1) & (a_0S_1) \end{pmatrix} = \begin{pmatrix} x_0 & x_1 \\ \epsilon(x_1S_2) & (x_0S_2) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

has a solution if $\delta = (\epsilon S_1)^{-1}$; then $a_0 = (x_0 S_2)$ and $a_1 = \epsilon(x_1 S_2)$. Thus

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} S_2 & 0 \\ 0 & S_2 R(\epsilon) \end{pmatrix}, \qquad U = P.$$

We have thus proved the following result:

THEOREM. *The algebra Ω_{ϵ,k_2} is isotopic with some Ω_{δ,k_1} if and only if $S_1 = S_2$ or S_2^{-1}, i.e., if and only if $k_1 = k_2$, or $k_1 = m - k_2$.*

An immediate corollary of the theorem is that if the algebras of [1] are defined with $S = T^k$, k arbitrary, not necessarily dividing m, new projective planes arise which are not isomorphic with the planes obtained when $k \mid m$.

The question of the structure of the collineation groups of the new planes arises and it should be pointed out that the methods of [1] are wholly applicable in the case of these more general algebras. The following lemma will be of assistance in studying the collineation groups (see [5] for a proof).

LEMMA. *If $(r, n) = 1$, $r < n$, q a prime power, then*

$$(1 + q^r, q^n - 1) = (1 + q, q^n - 1).$$

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INSTITUTE FOR DEFENSE ANALYSES,
PRINCETON, NEW JERSEY